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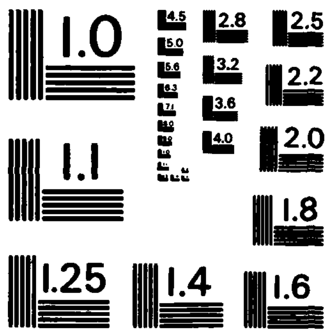

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A NOTE ON THE EFFECT OF IGNORING SMALL MEASUREMENT ERRORS IN PRECISION INSTRUMENT CALIBRATION

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Our focus is the simple linear regression model with measurement errors in both variables. It is often stated that if the measurement error in x is "small", then we can ignore this error and fit the model to data using ordinary least squares. There is some ambiguity in the statistical literature concerning the exact meaning of a "small" error. For example Draper and Smith (1981) state that if the measurement error variance in x is small relative to the variability of the true x's, then "errors in the x's can be effectively ignored", see Montgomery & Peck (1983) for a similar statement. Scheffe (1973) and Mandel (1984) argue for a second criterion, which may be informally summarized that the error in x should be small relative to (the standard deviation of the observed Y about the line)/(slope of the line). We argue that for calibration experiments both criteria are useful and important, the former for estimation of x given Y and the latter for confidence intervals for x given Y.

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Abstract

Our focus is the simple linear regression model with measurement errors in both variables. It is often stated that if the measurement error in x is "small", then we can ignore this error and fit the model to data using ordinary least squares. There is some ambiguity in the statistical literature concerning the exact meaning of a "small" error. For example, Draper and Smith (1981) state that if the measurement error variance in x is small relative to the variability of the true x's, then "errors in the x's can be effectively ignored", see Montgomery & Peck (1983) for a similar statement. Scheffe (1973) and Mandel (1984) argue for a second criterion, which may be informally summarized that the error in x should be small relative to (the standard deviation of the observed Y about the line)/(slope of the line). We argue that for calibration experiments both criteria are useful and important, the former for estimation of x given Y and the latter for confidence intervals for x given Y.

1. Introduction

There is substantial literature on the problem of precision instrument calibration, see for example Scheffe (1973), Rosenblatt and Spiegelman (1981) and Mandel (1984). We will focus on such calibration when fitting a straight line to a set of data in which the predictor x is measured with error.

We often advise on calibration problems, and recently we were asked to try to quantify what is meant by a "small" measurement error in x, with the idea that, if such error were small, we could safely ignore it and procede with ordinary least squares analysis. In trying to do this we realized that the literature is somewhat ambiguous, and in fact there are two distinct criteria used to decide when measurement error in x is small. For example, Draper and Smith (1981, page 124) state that if the measurement error variance in x is small relative to the variability of the true x's themselves, then "errors in the x's can be effectively ignored and the usual least squares analysis performed". This comment is echoed by Montgomery and Peck (1982, page 388). On the other hand, both Scheffe (1973, page 2) and Mandel (1984) use the criterion that we can safely ignore measurement error in x if its standard deviation is small relative to the ratio

$$\frac{\text{Standard deviation of measured Y about the line.}}{\text{Slope of the line}}$$

The authors were working in different contexts, so it is not surprising

that their criteria differ.

In this paper, we point out that for calibration experiments both criteria are useful. The criterion used by Draper and Smith is appropriate when the goal is estimation of intercept and slope based on the calibration data set, and then at the second stage for estimating the true value of x from a new observed Y. The criterion of Scheffe and Mandel addresses the issue of confidence intervals for estimating x from an observed Y. If the Draper and Smith criterion is satisfied while that of Scheffe and Mandel is not, the effect of ignoring the measurement error in x is essentially to cause larger confidence intervals for estimating the true value of x from new observed Y than is necessary.

Suppose that observed responses $\{Y_i\}$ are related linearly to the true working standards $\{x_i\}$ through the equation

(1.1) $$Y_i = \alpha + \beta x_i + \epsilon_i \ , \quad i = 1,2,\ldots N \ .$$

Here the deviations $\{\epsilon_i\}$ combine measurement errors in the response with equation or model error, and the $\{\epsilon_i\}$ are normally distributed with mean zero and common variance σ_ϵ^2.

Rather than observing the true working standards $\{x_i\}$, we observe

(1.2) $$X_i = x_i + v_i$$

where the measurement errors $\{v_i\}$ are assumed normally distributed with mean zero and variance σ_m^2. In the terminology of Fuller (1986), the equation (1.1) includes both equation error and response measurement error. From now on, when we speak of measurement error we will mean measurement error in the true $\{x_i\}$.

Assuming the working standards $\{x_i\}$ are measured without error, one would often procede as follows. First, perform the usual least squares analysis, which yields estimates $(\hat{\alpha}_L, \hat{\beta}_L, \hat{\sigma}_L)$. A new, independent observation Y_* is then made, and the goal is to estimate the value of x_* such that

$$E\, Y_* = \alpha + \beta\, x_* \, .$$

The maximum likelihood estimator is

$$\hat{x}_* = (Y_* - \hat{\alpha}_L)/\hat{\beta}_L \, . \tag{1.3}$$

For confidence intervals, the Working-Hotelling 100(1-α) % interval (Seber (1977)) for the unknown x_* is

$$I = \{x\colon Y_* \in \hat{\alpha}_L + \hat{\beta}_L\, x \pm t_\alpha\, \hat{\sigma}_L\, R(x)\} \, , \tag{1.4}$$

where t_α is the 1-α/2 percentage point of the t-distribution with N-2 degrees of freedom, and

$$R^2(x) = 1 + N^{-1}\{1 + (x-\bar{x})^2/s_x^2\} \, ,$$

where $\bar{x}$, s_x^2 are given by

$$\bar{x} = N^{-1} \sum_{1}^{N} x_i \, , \quad s_x^2 = N^{-1} \sum_{1}^{N} (x - \bar{x})^2 \, .$$

If the calibration is to be repeated, more complex confidence statements are available for those who wish to use them, see Scheffe (1973).

Draper and Smith's criterion for the severity of measurement error is

$$\frac{\text{measurement error variance in the } \{x_i\}}{\text{Variation of the } \{x_i\}} \doteq \frac{\sigma_m^2}{s_x^2} \, . \tag{1.5}$$

Scheffe and Mandel propose that the severity of measurement error depends

on the size of

(1.6) $\sigma_m^2/(\sigma_e/\beta)^2$.

In the next section we discuss the criteria (1.5)-(1.6) with regard to estimation and confidence intervals for x_* given an observed Y_*.

2. The Effect of Small Error

Let μ_x and s_x^2 denote the large sample mean and variance of the true working standards $\{x_i\}$, which are measured with variance σ_m^2. For large samples, the criterion (1.5) can be written as

(2.1) $\lambda = \sigma_m^2/s_x^2$,

and least squares estimates $(\hat{\alpha}_L, \hat{\beta}_L)$ converge in probability to $(\alpha + \lambda\, \mu_x \beta/(1+\lambda),\ \beta/(1+\lambda))$ respectively. By centering appropriately so that $\mu_x \simeq 0$, we see that the bias in least squares essentially depends on the size of λ in (2.1). When λ is small, for the purpose of estimation, the effect of ignoring measurement error in the true $\{x_i\}$ is slight.

Let us suppose that

(2.2) $\theta = \sigma_m^2/\sigma_e^2$

is known. From Kendall & Stuart (1961, pages 375-387), the maximum likelihood estimators of (α, β, σ) are given by

$$\hat{\alpha}_* = \bar{Y} - \hat{\beta}_* \bar{X}$$

$$\hat{\beta}_* = \frac{(S_Y^2 - \theta^{-1} S_X^2) + \{(S_Y^2 - \theta^{-1} S_X^2)^2 + 4\theta^{-1} S_{YX}^2\}^{\frac{1}{2}}}{2\, S_{YX}}$$

$$\hat{\sigma}_m = \theta^{-1}[S_X^2 - S_{YX}/\hat{\beta}_*],$$

where

$$S_x^2 = \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X})^2$$

$$S_Y^2 = \frac{1}{N} \sum_{i=1}^{N} (Y_i - \bar{Y})^2$$

$$S_{XY} = \frac{1}{N} \sum_{i=1}^{N} (X_i - \bar{X})(Y_i - \bar{Y}) \ .$$

It is known that the maximum likelihood estimator for σ is biased even in larger samples, and it is customary to make the correction

$$\hat{\sigma}_{m*} = 2\,\hat{\sigma}_m \ .$$

When λ is small, not only are the least squares estimators nearly the same as the maximum likelihood estimators, but in particular the least squares estimators are approximately unbiased as discussed previously. The story is considerably different when we turn to confidence intervals. Define

L_1 = length of the confidence interval for x_* given Y_* taking into account the measurement error in $\{x_i\}$.

L_2 = length of the confidence interval for x_* ignoring the measurement error in the $\{x_i\}$.

Then, for large enough sample sizes, in Appendix A we verify that when λ is (2.1) is small the ratio of these lengths is approximately

$$\frac{L_2}{L_1} \simeq (1 + (\sigma_m/(\sigma_e/\beta))^2)^{\frac{1}{2}} \ . \qquad (2.3)$$

Equation (2.3) verifies the criterion of Scheffe and Mandel that for confidence intervals, we can ignore measurement error in the working standards only if the measurement error has variance σ_m^2 small relative to $\sigma_\epsilon^2/\beta^2$. In the next section we provide an example where the criterion (1.5) mentioned by Draper & Smith is small but the Scheffe and Mandel criterion (1.6) is large.

3. An Example

In Table 1 we list a subset of the data investigated by Lechner, Reeve & Spiegelman (1982). It is not our purpose to provide a definitive analysis of these data. Rather, we use the data only to provide a means of exploring the effect of ignoring small measurement error, especially through the increased length ratio (2.3). We assume a straight line fit (1.1) to the data. We find that $\hat{\alpha}_L = -291.49$, $\hat{\beta}_L = 2346.64$ and $\hat{\sigma}_L = 1.64$. From discussion with the investigators it was thought that σ_m and σ_ϵ are of the same order of magnitude. However, since σ_ϵ is made up of both response measurement error and equation error, we decided to be rather conservative and set $\theta = 0.001$ in (2.2). We then used the rough approximations

$$\beta \simeq \hat{\beta}_* = 2346.64 \qquad \sigma_\epsilon = (1000)^{\frac{1}{2}}\sigma_m \simeq 0.0214$$

$$\sigma_m \simeq \hat{\sigma}_{m*} = 6.77 \times 10^{-4}$$

$$s_x^2 \simeq \underset{\text{observed X's}}{\text{sample variance of}} - \sigma_m^2 \simeq 0.57 \; .$$

obtaining therefore that $\lambda \leq 0.001$.

Clearly, λ is extremely small and, as expected, $\hat{\beta}_* \simeq \hat{\beta}_L$. This leads to the conclusion that for purposes of estimation, measurement error in the $\{x_i\}$ can be effectively ignored. However, the ratio of the lengths of the confidence intervals for x_* is approximately

$$L_2/L_1 \simeq (1 + \theta\ \beta^2)^{\frac{1}{2}} \simeq 74.2 \ .$$

This large ratio emphasizes our point that the definition of "small measurement error" must depend on whether one is interested in estimation or confidence intervals.

4. Conclusion

We have shown that, under the ideal conditions of a straight line model and a fairly large-sized working sample, ignoring measurement errors in x which are "small" relative to the usual estimation criterion (2.1) can result in calibration confidence intervals which are much larger than necessary. For confidence intervals, it is more sensible to judge measurement error size on the basis of both (1.5) and (2.3). Ignoring the measurement error in the true working standards $\{x_i\}$ will cause an increase in confidence interval length on the order of (2.3).

We finish by emphasizing that using measurement error techniques to obtain shorter calibration confidence intervals requires that equation (1.1) should hold. While least square confidence intervals can be very conservative in examples such as we have studied, they are more robust

against small model misspecifications. Small perturbations from the straight-line fit can significantly alter the coverage probabilities of the measurement error confidence interval I_1 without greatly affecting the coverage of the least squares intervals.

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Appendix A

In this appendix, we verify the approximation (2.3). While a precise large-sample analysis is routine, it is also notationally quite cumbersome. The essential ideas are perhaps easier to understand through the following heuristic analysis. Suppose that N is large and that λ in (2.1) is small.

Assuming that

(A.1) $$\sigma_m^2/\sigma_e^2 = \theta \text{ known,}$$

then maximum likelihood estimates $(\hat{\alpha}_*, \hat{\beta}_*)$ can be formed which are consistent for (α,β), see Fuller (1986). Under the assumption of small λ and large sample size N, we have

$$\hat{\alpha}_L \simeq \hat{\alpha}_* \simeq \alpha; \quad \hat{\beta}_L \simeq \hat{\beta}_* \simeq \beta ;$$

$$R(x) \simeq 1; \quad \hat{\sigma}_{m*} \simeq \sigma_L, \quad \hat{\sigma}_L \simeq (\sigma_e^2 + \beta^2 \sigma_m^2)^{\frac{1}{2}} .$$

Here $\hat{\sigma}_{m*}$ is the usual consistent estimate of σ_e under the assumption (2.2). Taking into account the measurement error in $\{x_i\}$ and using $(\hat{\alpha}_*, \hat{\beta}_*, \hat{\sigma}_{m*})$, within our heuristic framework the appropriate Working-Hotelling confidence interval for x_* is approximately

$$I_1 = \{x\colon Y_* \in \hat{\alpha}_* + \hat{\beta}_* \pm z_\alpha \hat{\sigma}_{m*}\} ,$$

where z_α is the $1-\alpha/2$ standard normal percentage point. The usual interval formed by ignoring measurement error is approximately

$$I_2 = \{x\colon Y_* \in \hat{\alpha}_L + \hat{\beta}_L x \pm z_\alpha \hat{\sigma}_L\} .$$

This latter interval is strictly appropriate not for x_* but rather for $X_* = x_* + v$. The length of the confidence interval I_1 taking into account

measurement error in $\{x_i\}$ is, for large samples, proportional to

(A.2) $$L_1 \simeq 2\, z_\alpha\, \sigma_e/\beta$$

while that for the usual least squares analysis is proportional to

(A.3) $$L_2 \simeq 2\, z_\alpha(\sigma_e^2 + \beta^2\, \sigma_m^2)^{\frac{1}{2}}/\beta\ .$$

The ratio of these lengths is, noting (A.1),

(A.4) $$\frac{L_2}{L_1} \simeq (1 + \{\sigma_m/(\sigma_{e/\beta})\}^2)^{\frac{1}{2}}\ .$$

TABLE

Pressure Tank Calibration Data

X	Y
2.08406	4599.3
2.08411	4600.1
2.27272	5044.1
2.27302	5042.7
2.27340	5044.3
2.46295	5488.1
2.46313	5486.5
2.65154	5931.1
2.65191	5931.5
2.65216	5931.6
2.84196	6379.7
2.84205	6379.8
3.03029	6817.5
3.03084	6817.3
3.03108	6817.9
3.22096	7266.4
3.22114	7268.3
3.40919	7709.3
3.40977	7709.6
3.40994	7710.5
3.59999	8155.5
3.60028	8157.5
3.78805	8597.2
3.78871	8599.1
3.78883	8600.3
3.97893	9048.4
3.97932	9047.8
4.16693	9484.2
4.16762	9486.6
4.16781	9486.6
4.35790	9935.5
4.35825	9938.3
4.54579	10377.0
4.54660	10378.6